Math 434 Assignment 1

Due March 15

Assignments will be collected in class.

1. We defined the ordered pair (a, b) as $\{\{a\}, \{a, b\}\}$.
   (a) Prove, using the axioms of ZFC and stating which you are using, that (a, b) exists.
   (b) Prove that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.
2. We defined addition on the natural numbers. Using this definition, prove:
   (a) for all $a \in \mathbb{N}$, $0 + a = a + 0 = a$.
   (b) for all $a, b \in \mathbb{N}$, $a + b = b + a$.
3. Formally define the multiplication function $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and prove that it is:
   (a) a set,
   (b) a function,
   (c) has domain $\mathbb{N} \times \mathbb{N}$.
4. Prove that if $(A, \leq)$ is a well-order, and $B \subseteq A$, then $(B, \leq)$ is also a well-order.
5. Let $(A, \leq_A)$ and $(B, \leq_B)$ be linear orders. Define the lexicographic order $\leq_{lex}$ on $A \times B$ as follows:
   $$(a, b) \leq_{lex} (a', b') \iff a <_A a', \text{ or } a = a' \text{ and } b \leq_B b'.$$
   Prove:
   (a) $\leq_{lex}$ is a linear order.
   (b) if A and B are well-orders, then so is $\leq_{lex}$.
6. Use Lemma 2.6 (and the ideas from its proof) to prove Corollary 2.7 and Lemma 2.8:
   (a) Prove that well-orders are rigid: If $(A, \leq)$ is a well-order and $f : A \to A$ is an isomorphism, then f is the identity.
   (b) Prove that a well-order is not isomorphic to any proper initial segment of itself.